- **Mutually Exclusive:**

  2 events cannot happen at the same time

  ↳ coin tossing: H & T cannot both exists

- **Collectively exhaustive**

  ↳ The events include everything

- For a equally likely, mutually exclusive, collectively exhaustive event:

$$P(E) = \frac{\text{\# of outcome favorable to } E}{\text{\# of total outcomes}}$$

↳ ie: the entire sample space

- **Sample Space**

  ↳ if equally likely: uniform

  ↳ if not equally likely: non-uniform

- For any sample space,

$$P(E) = \sum P(\text{events favorable to } E)$$

- **Probability Theorems:**

$$P(A \cap B) = P(A) \times P(A|B)$$

$$P(A) = \frac{N(A)}{N}$$

$$P(A|B) = \frac{N(A \cap B)}{N(A)}$$

For independent events

↳ ie: the outcome of A won't affect B

$$P(A \cap B) = P(A)\,P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

↳ if $P(A \cup B) = P(A) + P(B)$

↳ A & B are mutually exclusive

Baye's Formula:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

Permutation: $P_{(n,r)} = {}_nP_r = \dfrac{n!}{(n-r)!}$

↳ # of ways to pick r out of n w/ order

$$C_{(n,r)} = {}_nC_r = \frac{n!}{(n-r)!\,r!}$$

↳ # of ways to pick r out of n w/o order

Binomal: Expansion $(x+a)^n = \sum_{i=0}^{i=n} ({}_nC_i)\,x^i a^{n-i}$

• **Mean:** → average

$\bar{x} = \mu_x = E(x) = \sum_i x_i p_i$ ← weighted avg.

• **Varience** → how far a number is away from mean.

$Var(x) = \sigma_x^2 = \sum_i (x_i - \mu)^2 f(x_i)$

$= E(x^2) - [E(x)]^2$

• **Standard deviation** → the "spread" of the probability distribution

$\sigma = \sqrt{\sigma^2}$

• **Cumulative Distribution**

$F(x<x_i) = \sum_{j=1}^{i} f(x_j)$

↳ from $x=1$ to the x_i

• For a **continuous distribution:**

$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$ ← probability function

$\sigma_x = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

$F(x<a) = \int_{-\infty}^{a} f(x)\, dx$

$\int_{-\infty}^{\infty} f(x)\, dx = 1$

Binomial distribution:

for 2 outcomes, where:

p = prob. of sucess

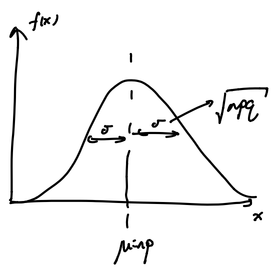
q = prob. of failure $= 1-p$

$\boxed{f(x) = {}_nC_x\, p^x q^{n-x}}$

$\mu_x = \langle x \rangle = np$ ← as $n\uparrow$, $\langle x \rangle \uparrow$

$\sigma^2 = npq$

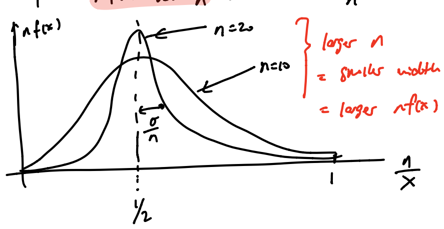
$\sigma = \sqrt{npq}$



as $n\uparrow$ distribution becomes wider

$\therefore \sigma$ is bigger → spread is wider

• width of a $nf(x)$ vs. $\frac{x}{n}$ distribution $= \frac{\sigma}{n} = \frac{\sqrt{npq}}{n}$



Normal Distribution

$$f(x) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{(x-\mu_0)^2}{2\sigma^2}}$$

Can approximate binomial for a large n

$$(nCx)\, p^x q^{n-x} \approx \frac{1}{\sqrt{2\pi \underbrace{npq}_{\sigma^2}}} e^{\frac{-(x-\overbrace{np}^{\mu})^2}{2\underbrace{npq}_{\sigma^2}}}$$

<u>Z-score:</u>

$$Z = \frac{x-\mu}{\sigma}$$

$$z = \frac{a-\mu}{\sigma}$$

$$F(x < x=a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}}\, dy$$

<u>Note:</u> $x = Z\sigma + \mu$

↳ z is a measure of how far x is from μ

Central Limit Theorem:

Σ of independent large population of x is approximately normally distributed

Population: a complete collection of individuals under consideration

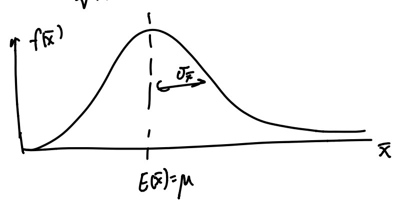
Sample: A portion of the population <u>randomly</u> chosen

Sample average:

$$\bar{x} = \frac{1}{n}\sum_i x_i \Rightarrow E(\bar{x}) = \mu$$

Standard deviation of $\bar{x}$:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \qquad \sigma_{\bar{x}}^2 = Var(\bar{x}) = \frac{\sigma^2}{n}$$



Estimate population variance (σ^2)

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\bar{x}_i - \mu)^2$$

$$↳ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{\sum_{i=1}^{n}(\bar{x}_i-\mu)^2}{n(n-1)}} = \sigma_m$$

$$\int_{\mu-\sigma}^{\mu+\sigma} f(x)\, dx = 68\%$$

$$\& \int_{\mu-\sigma_m}^{\mu+\sigma_m} f(\bar{x})\, dx = 68\%$$

$$\mu - \sigma_m \leq \bar{x} \leq \mu + \sigma_m \rightarrow \bar{x} - \sigma_m \leq \mu \leq \bar{x} + \sigma_m$$

Error Analysis

$x = \bar{x} \pm \sigma_{\bar{x}} \qquad y = \bar{y} \pm \sigma_{\bar{y}}$

Give $w(x, y)$

$$W = \bar{w} \pm \sigma_{\bar{w}}$$

$$\sigma_{\bar{w}} = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2\Bigg|_{\substack{x=\bar{x}\\y=\bar{y}}} \sigma_{\bar{x}}^2 + \left(\frac{\partial w}{\partial y}\right)^2\Bigg|_{\substack{x=\bar{x}\\y=\bar{y}}} \sigma_{\bar{y}}^2}$$

$$\bar{w} = w(\bar{x}, \bar{y})$$

Curve Fitting

Least Square fit:

$$S(A,B) = \sum_{j=1}^{n} [\underbrace{y_j}_{\text{actual value}} - \underbrace{(A+Bx_j)}_{\text{estimated value using } y=A+Bx}]^2$$

Good estimation has $S(A,B)$ at minimum

Give $y = A + Bx$

$$A = \bar{y} - B\bar{x}$$

$$B = \frac{S_{xy}}{S_x^2}$$

sample covarience

sample verience

$$S_{xy} = \frac{1}{n-1} \sum_j (x_j - \bar{x})(y_j - \bar{y})$$

$$S_{x^2} = \frac{1}{n-1} \sum_j (x_j - \bar{x})^2$$

Some others.

$$E(xy) = E(x)\,E(y)$$

$$E(ax+by) = aE(x) + bE(y)$$

• Random Motion

- For random motion of N steps...

  ↳ $\langle \vec{R} \rangle = \emptyset$

  $\langle \vec{R}^2 \rangle = Nb^2$, b = step size

  ↳ $\underbrace{\sqrt{\langle \vec{R}^2 \rangle}} = \sqrt{N}\, b$

  root mean square of $\vec{R}$

- Since N is not very measurable, we can connect N with t:

  ↳ t = time for N steps

  time for N steps

  $\therefore \langle \vec{R}_N^2 \rangle = \frac{t}{\Delta t} b^2$

  time per N steps

  $= 2 \frac{b^2}{2\Delta t} t$

  $\langle \vec{R}_N^2 \rangle = 2Dt$, $D = \frac{b^2}{2\Delta t}$ = diffusion constant (in 1D)

• Kinetic Energy

- Temperature is directly proportional to the kinetic energy of molecule

  $T = \frac{m \langle v^2 \rangle}{3 K_B}$

  $\langle KE \rangle = \frac{3}{2} K_B T$ in 3D

  $\begin{cases} \langle KE \rangle = \frac{1}{2} K_B T & \text{in 1D} \\ \sqrt{\langle V_x^2 \rangle} = \sqrt{\frac{K_B T}{m}} & \text{in 1D} \end{cases}$

• Large Molecule in Water

- Frictional Force

  ↳ $-\int \frac{dx}{dt}$ = Friction/Restraining Force

  ↳ Viscous Friction Coefficient

- Measure ζ

$$\vec{F}_{TOT} = \vec{F}_{ext} - \vec{F}_{fric}$$

$$\Rightarrow \frac{m\,dx^2}{dt^2} = F_{ext} - \zeta \frac{dx}{dt}$$

$$\frac{m\,dv_x}{dt} = F_{ext} - \zeta v_x(t)$$

if $v_x(0) = 0$

$$\Rightarrow v_x(t) = \frac{F_{ext}}{\zeta}\left(1 - e^{-\frac{\zeta t}{m}}\right)$$

at $\lim_{t\to\infty} v_x(t) = \frac{F_{ext}}{\zeta}$ = terminal velocity

$$\therefore \zeta = \frac{F_{ext}}{v_x(\infty)} = \frac{F_{ext}}{v_{terminal}}$$

- Einstein's Relation

$$\zeta D = K_B T$$

energy converted into heat

(disspation)

varience of particles' position

(fluctuaction)

- Diffusion

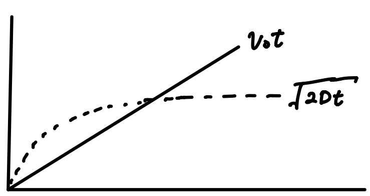
↳ Effective for small molecules or short distances

$$\langle x^2 \rangle = 2Dt$$

↳ Compare with Directed Transport

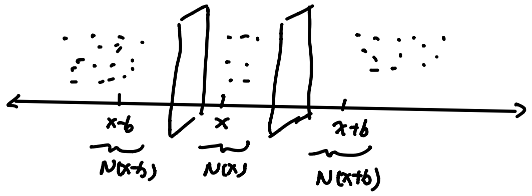
↳ $x = v_0 t$,

$$v_0 \simeq \frac{10 \text{ body length}}{\text{sec}} = \frac{10L}{s}$$



at $t = \frac{0.01D}{L^2}$, directed transport is faster

- Diffusion from Mascropic Perspective



Let J_x be the flux

↳ net particle to right in a unit time per area

$$J_x = \frac{1}{2}\frac{[N(x+b)-N(x)]}{A\Delta t}$$

concentration

$$= -D\,\frac{n(x+b)-n(x)}{b}$$

$$\lim_{b\to 0} J_x = -D\left(\frac{\partial n}{\partial x}\right)_y$$

$$\therefore \boxed{J_x = -D\frac{\partial n}{\partial x}}$$

Fick's First Law of Diffusion

↳ -ve sign mean flux is in the direction of high concentration to low concentration

In 3D:

$$\vec{J} = -D\,\vec{\nabla} n(x,y,z)$$

→ In polar:

$$J_r = -D\frac{\partial n}{\partial r} = J\cdot\hat{r}$$

→ Total flux into a perfect absorber cell

$$I = 4\pi D\, n(\infty) R$$

Metabolic constant/rate coefficient

- Metabolic needs $= \frac{4}{3}\pi R^3 M$

↳ ∴ To survive:

$$\frac{4\pi}{3}R^3 M \leq 4\pi D\, n(\infty) R$$

$$\boxed{R \leq \sqrt{\frac{3D\,n(\infty)}{M}} = R_{max}}$$

↳ Max cell size

• Fick's 2nd Diffusion Equation

$$\frac{\partial n(x,t)}{\partial t} = \frac{\partial J(x,t)}{\partial x}$$ By conservation of particle

$$\rightarrow \frac{\partial n}{\partial t} = -\frac{\partial}{\partial x}\left(-D\frac{\partial n}{\partial x}\right) = D\frac{\partial^2 n}{\partial x^2} = \frac{\partial n}{\partial t}$$

↳ This is a density gradient

↳ Flows from high density to low density

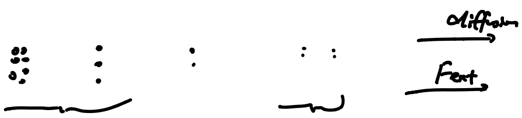
• If we define flux as...

$$J_x = n v_x = n\frac{F_{ext}}{\zeta}$$

Compare with:

$$J_x = \frac{n F_{ext}}{\zeta} \leftrightarrow -D\frac{\partial n}{\partial x} = J_x$$

$$F_{ext} \longleftarrow -k_B T \frac{1}{n}\frac{\partial n}{\partial x}$$



$\frac{\partial n}{\partial x} \neq 0$ $\qquad$ $\frac{\partial n}{\partial x} = 0$

$F_{ext} = 0$ $\qquad$ $F_{ext} \neq 0$

∴ J_x is the same if

$$F_{ext} = -k_B T \frac{1}{n}\frac{\partial n}{\partial x}$$

→ Particles are driven by concentration gradient

↳ We can consider the gradient as an entropic force that moves the particle

→ Can also see $\frac{n F_{ext}}{\zeta}$ as a contribution to J_x:

$$J_x^{tot} = -D\frac{\partial n}{\partial x} + \frac{n F_{ext}}{\zeta}$$

→ In equilibrium, $J_x^{tot} = 0$

↳ no net flow of particle

$$\therefore -\frac{\partial n}{\partial x} + n\frac{F_{ext}}{\zeta} = 0$$

$$\frac{n(x_2)}{n(x_1)} = \frac{e^{(F_{ext}x_2)/k_B T}}{e^{(F_{ext}x_1)/k_B T}}$$

Let $x_1 = 0$, $x_2 = x$

$$\therefore n(x) = n(0)e^{F_{ext}x/k_BT}$$

- Chemical Potential at x:

$$\mu(x) = \underbrace{F_{ext}x}_{\text{energetic}} + \underbrace{k_BT\ln(x)}_{\text{entropic}}$$

- Nernst Potential

$$(V_2 - V_1) = \frac{1}{e}k_BT\ln\left(\frac{n_1}{n_2}\right)$$

$$\Delta V = \frac{1}{e}k_BT\ln\left(\frac{n_1}{n_2}\right)$$

- Biological Elasticity

↳ $S = k_B\ln(\underbrace{w(x)}_{})$

$S \downarrow$ entropy

$w(x) \downarrow$ # of desired outcomes

$$F = \frac{d}{dx}(U + TS)$$

$$= \frac{k_BT}{Nb^2}x$$

$$F = K_{entropic}\,x$$

↳ entropic spring

↳ As T↑, length ↓

- Free Energy = $E - TS$

↳ At a fixed temp, a process is spontaneous iff it reduces the free energy

↳ no spontaneous change if free energy is already at a min

↳ Free energy minimization

- S_{TOT} maximization = Free Energy$_{system}$ Minimization

- In equilibrium,

$$\frac{\partial\, \text{Free } E}{\partial x} = 0$$

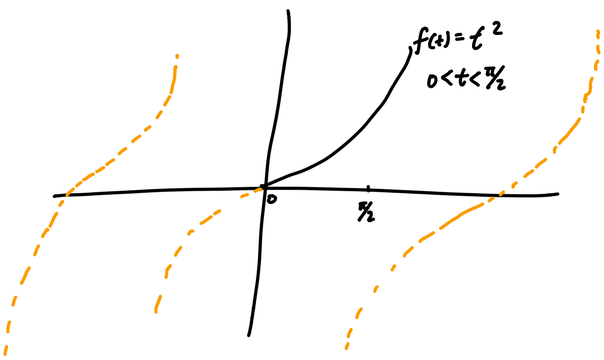
• Fourier Series

$$f(t) = \frac{a_0}{2} + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \cdots + b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \cdots$$

$$\begin{cases} a_n = \frac{2}{\pi} \int_{-T/2}^{T/2} f(t) \cos(n\omega t)\, dt, & n = 0, 1, 2, 3, \ldots \\ b_n = \frac{2}{\pi} \int_{-T/2}^{T/2} f(t) \sin(n\omega t)\, dt, & n = 1, 2, 3, \ldots \end{cases}$$

$$\omega = \frac{2\pi}{T}$$

- If $f(t)$ is an odd function, $a_n = 0$
  - ↳ There is only sin in final answer
- If $f(t)$ is an even function, $b_n = 0$
  - ↳ There is only cos in final answer
- Fourier converges to $f(t)$ at all points, except at discontinuity
  - ↳ Gibb's phenomenon
    - ↳ Converge to midpoint b/w 2 discontinuity
- To model non periodic function
  - ↳ over an interval:
    - ↳ expand the interval to make it periodic



- As position (x) and wavelength ($\lambda = 2l$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{\lambda}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{\lambda}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\begin{cases} a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx \\ b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \end{cases}$$

• Periodic Function in Complex Representation

Given:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We have:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega t}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega t} dt$$

Similarly:

$$f(x) = \sum_{x=-\infty}^{\infty} C_n e^{in\pi x/\ell}$$

$$C_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell} dt$$

• Fourier Transformation

For a non-periodic, yet large T:

• Given $f(t)$, we have:

$$\begin{cases} f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \\ F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \end{cases}$$

Fourier transform of $f(t)$

or:

$$\begin{cases} f(x) = \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \\ F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \end{cases}$$