- **Equation of a line**

  - $$\vec{r}(t) = \underbrace{\vec{a}}_{\text{direction vector}} t + \underbrace{\vec{b}}_{\text{a point on line}}$$

  - $$\begin{cases} x(t) = t a_x + b_x \\ y(t) = t a_y + b_y \\ z(t) = t a_z + b_z \end{cases}$$

- **Equation of a plane**

  - $$\vec{x} = \underbrace{s\vec{u} + t\vec{v}}_{\text{2 directions}} + \underbrace{\vec{b}}_{\text{one point}}$$

  - $$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{b}$$

    $$ax + by + cz = b$$

    $$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- **Trajectory of particles**

  - collide when $\vec{r}_1(t) = \vec{r}_2(t)$

  - Velocity vector, $\vec{r}\,'(t)$ is always tangent to $\vec{r}(t)$

  - Angle of collision:

    $$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \, \|\vec{v}_2\|}$$

- **Arc Length**

  $$L = \int_a^b \|\vec{r}\,'(t)\| \, dt$$

## Line Integral

How much a function accumulates along a line.

- $$\int_c f \, dl = \int_c f \, \|\vec{r}\,'(t)\| \, dt$$

  For scalar function

- $$\int_c \vec{F} \cdot d\vec{l} = \int_c \vec{F} \cdot \vec{r}\,'(t) \, dt$$

  For vector function

## Level curve

- given $f(x,y) = z$, where $z \in \mathbb{R}$.
- traces: fix any $x, y,$ or $z$

## Quadratic Surfaces

- sphere: $x^2 + y^2 + z^2 = R^2$
- eclispe: $(ax)^2 + (bx)^2 + (cx)^2 = R^2$

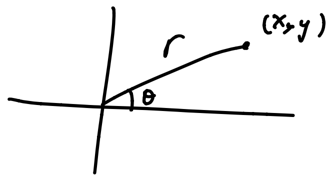
- Equations for level curves:

  - $x^2 + a = y \longrightarrow$ parabola
  - $x^2 - y^2 = a \longrightarrow$ hyperbola
  - $x^2 + y^2 = a^2 \longrightarrow$ circle
  - $(ax)^2 + (bx)^2 = a^2 \longrightarrow$ eclipse

## Coordinate System

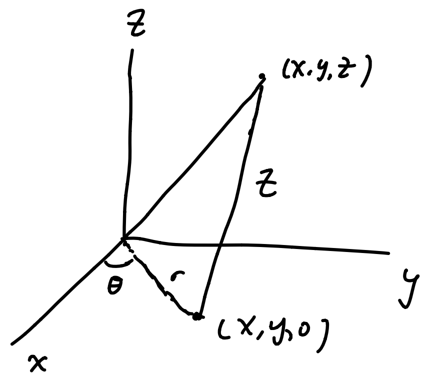
- Polar

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$




- Cylindrical

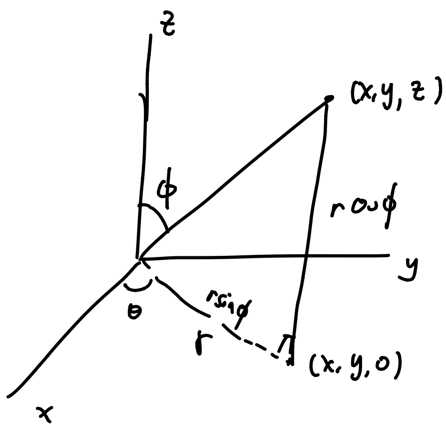
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$




- spherical

$$\begin{cases} x = r\cos\theta\sin\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}$$

$$x^2+y^2+z^2 = r^2$$



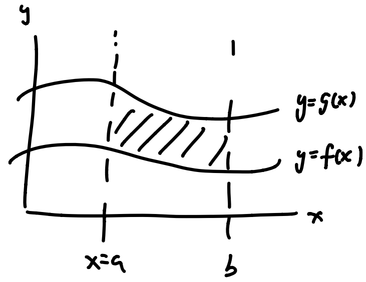

## Double Integral

How much a function accumulates over a given area.

$$V = \iint_A f(x,y)\,dA \quad , \quad \text{if } f(x,y) = 1, \quad A = \iint_A dA$$

- $$\int_{x=a}^{b}\int_{y=c}^{d} f(x)\,g(y)\,dx\,dy = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

Fubini's Theorem




$$\int_{x=a}^{b} \int_{y=f(x)}^{g(x)}$$

Cannot apply Fubini's theorem here.

- Polar:

$$dA = r\,dr\,d\theta$$

- Moment of Inertia:

$$I = \int_A r^2\,dm = \iint_A r^2 \rho\,dA$$

- Triple Integral

Amount of $f$ that accumulates over $E$.

$$\iiint_E f\,dv$$

- Cartesian

$$dv = dx\,dy\,dz$$

- Cylindrical

$$dv = r\,dr\,d\theta\,dz$$

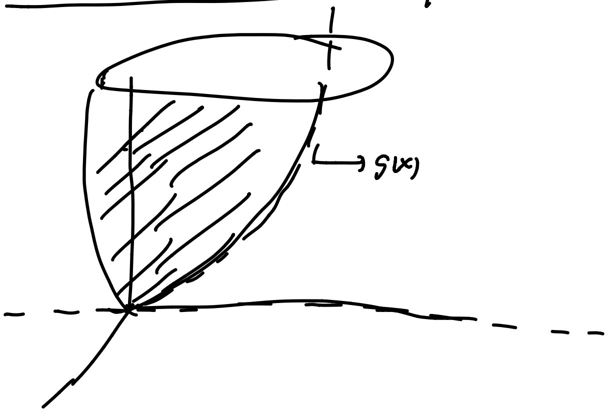
- Sphereical

$$dv = r^2 \sin\phi\,dr\,d\theta\,d\phi$$

- Moment of Inertia

$$I = \int r^2\,dm = \iiint_E r^2 \rho\,dv$$

## Triple Integral over arbitrary area




Use function as boundaries at appropriate area

## Partial Derivatives

- $\frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x} = f_x$ → hold $y$ constant

$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

- Claraut's Theorem

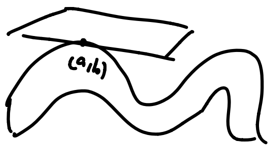
If $f(x,y)$ is cont. and defferentiable on some regions $D$, then

$f_{xy}(x,y) = f_{yx}(x,y)$ over $D$

## Tangent plane of a Multivariable Function

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

↳ Equation of tangent plane at $f(a,b)$




Use tangent plane to approximate a function near $f(a,b)$

## • Differential

for $z = f(x,y)$

$$dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy$$

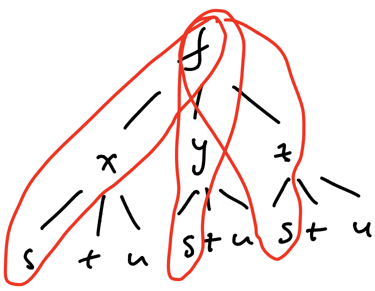
Use to approximate small changes in $z = f(x,y)$

## • Chain Rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Use a tree diagram

↳ Given $f(x,y,z)$, and $x(s,t,u)$, $y(s,t,u)$, $z(s,t,u)$




$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dy}{ds}$$

## • Change of Variables

$$\iint_D f(x,y)\, dx\, dy$$

Let $x = x(u,v)$

Let $y = y(u,v)$

$$= \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

change of variable from $x,y \to u,v$

$$\cdot \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{array}{cc} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} & \begin{array}{c} \partial x \\ \partial y \end{array} \\ \begin{array}{cc} \frac{1}{\partial u} & \frac{1}{\partial v} \end{array} & \end{array}$$

- $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ $\begin{matrix} \partial x \\ \partial y \\ \partial z \end{matrix}$

$\frac{1}{\partial u}$ $\frac{1}{\partial v}$ $\frac{1}{\partial w}$

- Directional Derivatives

- $D_{\vec{u}} f = a f_x(x,y) + b f_y(x,y)$

  $\vec{u} = (a, b)$ and $\|\vec{u}\| = \sqrt{a^2+b^2} = 1$

  $\vec{u} = (\cos\theta, \sin\theta)$

- $D_{\vec{u}} f = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$, $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$

  ↳ $\vec{\nabla} f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$

$\therefore$ $\boxed{\begin{aligned} D_{\vec{u}} f &= \vec{\nabla} f(x,y) \cdot \vec{u} \\ &= \|\vec{\nabla} f(x,y)\| \, \|\vec{u}\| \cos\theta \end{aligned}}$

- If $\vec{u}$ is in the same direction of $\vec{\nabla} f$

  ↳ Max increase in $f$, with magnitude $\|\vec{\nabla} f\|$

- If $\vec{u}$ is perpendicular to $\vec{\nabla} f$

  ↳ Level curves

- $\vec{\nabla} r = \hat{r}$

- **Vector Field**

mapping from $\mathbb{R}^n \to \mathbb{R}^n$

$$\vec{F}(x,y,z) = \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z \end{bmatrix}$$

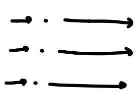
- **Gradient Field**

$$\underbrace{\vec{F}}_{\text{Force Field}} = -\vec{\nabla} \underbrace{f}_{\text{Potential}} \xrightarrow{\text{In general}} \vec{F} = \vec{\nabla} f$$

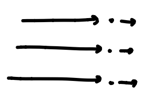
- **Divergence:**

- $$\vec{\nabla} \cdot \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z \end{bmatrix}$$

- Tells us the net flux out one point.



+ve divergence

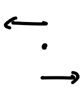


-ve divergence

- **Curl:**

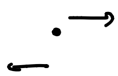
- $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Tells us the "spinning" of the field. → +ve if CCW.



+ve Curl

→ $\vec{\omega} = ||\vec{\omega}|| \hat{z}$



-ve curl

$\vec{\omega} = -||\vec{\omega}|| \hat{z}$

- Path Independence

  - A vector field is conservative iff:

$$\boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$$ curl free

In 2D: $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$

- If it is conservative, then it is path independent, and a potential $f$ exists:

$$\int_c \vec{F} \cdot d\vec{r} = \int_{t_i}^{t_f} (\vec{\nabla} f) \cdot d\vec{r} = \boxed{f(t_f) - f(t_i)}$$

- Find potential given a conservative $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$

$$\int F_1 \, dx = g(x,y,z) + h(y,z) = f$$

verify using other components:

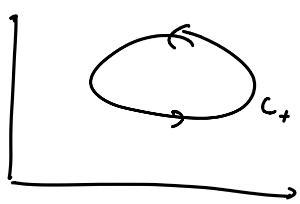
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y} g(x,y,z) + \frac{\partial}{\partial y} h(y,z) = F_2$$

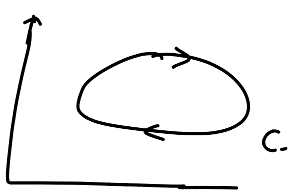
$$\rightarrow h(y,z) = i(y) + j(z)$$

⋮

- Green's Theorem

  - Convention: path is always CCW. for +ve.




$$\oint_{C_+} \vec{F} \cdot d\vec{r} = - \oint_{C_-} \vec{F} \cdot d\vec{r}$$

- $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$  Green's Theorem


$\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

↳ if the domain of $\vec{F}$ is simply connected.

- Surface Integral

$S = \iint_S ds$ → Integrate over a surface.

- spherical:

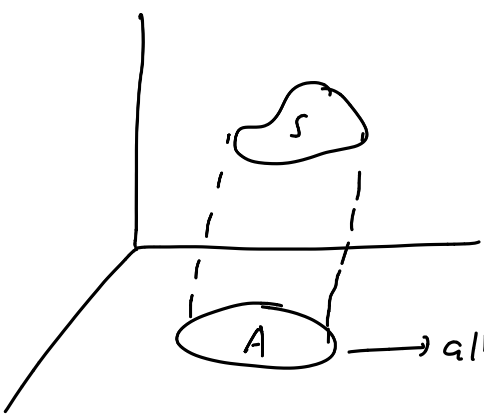
$ds = r^2 \sin\phi \, d\theta \, d\phi$

- Cylindrical:

$ds = r \, d\theta \, dz$

- General surface: $z = f(x,y)$

$ds = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy \longrightarrow \int_S ds = \int_A \sqrt{1+f_x^2+f_y^2} \, dA$




→ all the area of S above A.

- Surface Integral On vector Fields

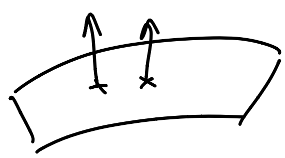
Flux through a surface $S$:

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S (\vec{F} \cdot \hat{n})\, ds$$

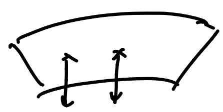
- Orientability:

  ↳ not orientable if $\hat{n}$ changes at different places

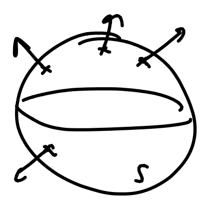
- In general, 2 choices of $\hat{n}$



and



- For closed surfaces (ie: sphere), $\hat{n}$ points outward.




$\hat{n} = \hat{r}$

$$\oiint_S \vec{F} \cdot d\vec{s} = \text{flux of } \vec{F} \text{ out of a closed } S$$

- For a general surface $S$, $\hat{n}$ is given by:

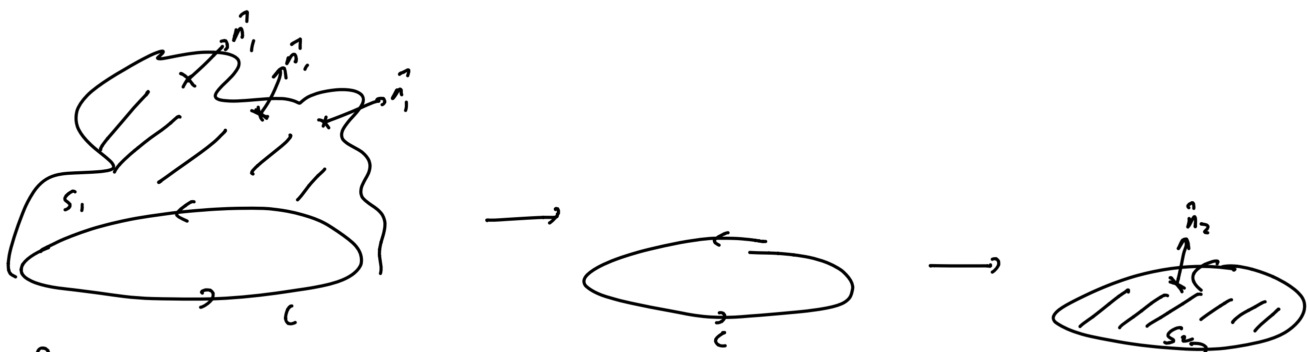
$$\hat{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

- Stoke's Theorem

  ↳ Green's theorem's more generalized form.

$$\oint \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, ds$$

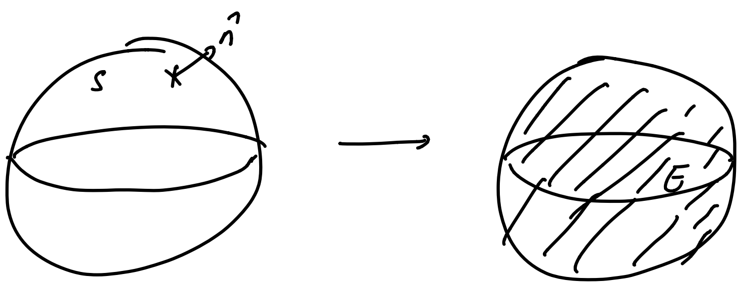
  ↳ We can have:




$$\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}_1 \, ds \qquad \int_C \vec{F} \cdot d\vec{r} \qquad \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}_2 \, ds$$

- Divergence Theorem

$$\oiint_S \vec{F} \cdot d\vec{s} = \iiint_E (\vec{\nabla} \cdot \vec{F}) \, dv$$



- Fields in $\frac{\vec{r}}{r^2}$

↳ undefined at $r=0$ $\longrightarrow$ $\vec{\nabla} \cdot \vec{F} = 0$ everywhere, except $r=0$

↳ Delta Function:

1D: $$\delta(x) = \begin{cases} \infty, & x=0 \\ 0, & x \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

3D: $$\delta^3(\vec{r}) = \begin{cases} \infty, & \vec{r}=0 \\ 0, & \vec{r} \neq 0 \end{cases} \quad \text{and} \quad \iiint_E \delta^3(\vec{r})\, dV = 1$$

Define: $$\boxed{\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^2}\right) = 4\pi\, \delta^3(\vec{r})}$$

↳ can use to approximate charge distribution

- Extreme Values

- Critical points: $f_x(a,b) = f_y(a,b) = 0$ or undefined

- Classify critical points

$$\boxed{D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2}$$

if:

| | | | |
|---|---|---|---|
| $D>0,\ f_{xx}>0$ | $\longrightarrow$ | minimum | concave up |
| $D>0,\ f_{xx}<0$ | $\longrightarrow$ | max | concave down |
| $D<0$ | $\longrightarrow$ | saddle | |
| $D=0$ | $\longrightarrow$ | inconclusive | |

- Lagrange Multipliers

maximum of $f$ given a boundary of $g$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

- Find 3 expressions for $\lambda$, then use that to find relationships b/w $x, y, z$
- Plug $x, y, z$ into $g$ and solve