Basic Set Notations

- $x \in A$ — x is an element of A
- $x \notin A$ — x is not an element of A
- $x \subseteq B$ — x is a subset of A
- $B \supseteq x$ — B is a superset of x
- $A = B$ — iff $A \subseteq B \wedge B \subseteq A$
- $A \subsetneq B$ — if $A \subseteq B \wedge A \neq B$
- $A \not\subseteq B$ — if A is not a subset of B
- $A \cup B$ — $\{x \mid x \in A \overset{\text{or}}{\vee} x \in B\}$
- $A \cap B$ — $\{x \mid x \in A \wedge x \in B\}$

Numbers

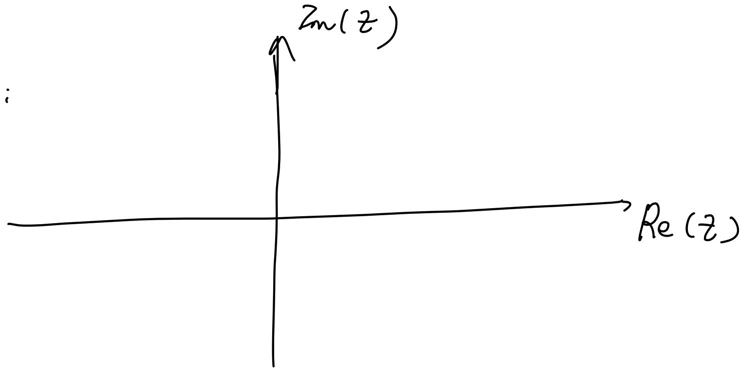
$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

$$\mathbb{Q}^c \subsetneq \mathbb{R}$$

Complex Numbers

$$z = a + bi$$

Argand Diagram:



Sum: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$

Scalar Multiplication: $cz_1 = ca + cbi$

Multiplication: $z_1 \cdot z_2 = (a_1 + b_1 i) \cdot (a_2 + b_2 i)$

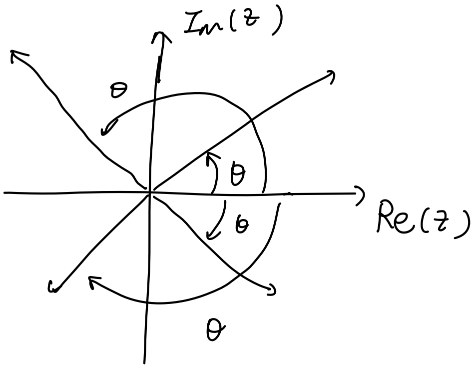
Conjugate: $\bar{z} = a - bi$

Modulus: $|z| = \sqrt{a^2 + b^2} = z\bar{z}$

Quotient: $\dfrac{z_1}{z_2} = \dfrac{z_1}{z_2} \dfrac{\bar{z}_2}{\bar{z}_2}$

Polar forms

$$\operatorname{Arg}(z) = \theta \quad , \quad -\pi < \operatorname{Arg}(z) \leq \pi$$



$$z = |z|\left(\cos(\theta) + i\sin(\theta)\right)$$

$$= |z| \operatorname{cis}(\theta)$$

$$\bar{z} = r \operatorname{cis}(-\theta)$$

$$z_1 z_2 = |z_1||z_2| \operatorname{cis}(\theta_1 + \theta_2)$$

$$z^2 = |z|^2 \operatorname{cis}(2\theta)$$

$$(\cos\theta + i\sin\theta)^n = \operatorname{cis}(n\theta)$$ → De Moivre's Formula

$$e^{i\theta} = \operatorname{cis}(\theta)$$ → Euler's Formula

$$|z_1 + z_2| \leq |z_1| + |z_2|$$ → triangle inequality

Vectors

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} = \vec{u} \cdot \vec{u}$$

Linear Combination

for $\vec{v}_1, \dots, \vec{v}_n$, $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ is a linear combination

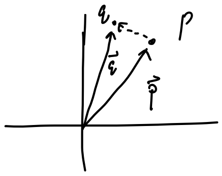
↳ $\{\vec{v}_1, \dots, \vec{v}_n\}$ is <u>linearly independent</u> iff $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$ implies that all $c_1, c_2, \dots, c_n = 0$

↳ if at least one c_i is non-zero, the set is <u>linearly</u> <u>dependent</u>

Vector Equation of a Line

$$\vec{x} = \vec{p} + k\vec{d}, \quad k \in \mathbb{R}, \quad \vec{d} \neq \vec{0}$$

Directed Line Segment



$$\overrightarrow{PQ} = \vec{q} - \vec{p}$$

If $\overrightarrow{PQ} = \overrightarrow{RS}$, we say $\overrightarrow{PQ}$ is <u>equivalent</u> to $\overrightarrow{RS}$

(Not equal!)

Vector Equation of a plane

$$\vec{x} = \vec{p} + t\vec{v} + s\vec{u}$$

↳ $\vec{u}$ & $\vec{v}$ must be <u>Linearly</u> <u>independent</u>

Span

A collection of all linear combination

if $S = \{\vec{v}_1, \dots, \vec{v}_k\}$

$$\text{Span}(S) = \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

If a set from S is linearly dependent with others, we can remove it, and the span remains the same.

Basis

A linearly independent spanning set

if $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$:

1) $\text{Span}(B) = \mathbb{R}^n$

2) B is linearly independent

$\Rightarrow$ There are infinetly many basis for $\mathbb{R}^n$

$\Rightarrow$ Standard basis for $\mathbb{R}^2$: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

Dot Product

$$\vec{p} \cdot \vec{q} = p_1 q_1 + p_2 q_2 + \dots + p_n q_n = \|\vec{p}\| \|\vec{q}\| \cos(\theta)$$

If $\vec{p}, \vec{q} \in \mathbb{C}^2$:

$$\vec{p} \cdot \vec{q} = p_1 \overline{q_1} + p_2 \overline{q_2}$$

Two vectors are orthogonal if their dot product $= 0$

Cauchy-Schwartz Inequality: $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

Unit Vector : $\hat{x} = \dfrac{\vec{x}}{\|\vec{x}\|}$

Scaler Equation of a plane

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

- Two planes are parallel iff the normal vectors are Scalar multiple
- Two planes are orthogonal iff the normal dot product is $\emptyset$

Cross Product

A vector that is perpendicular to both

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\vec{u} \times \vec{v} = \begin{bmatrix} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \\ -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \\ \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \end{bmatrix} \qquad \begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{matrix}$$

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

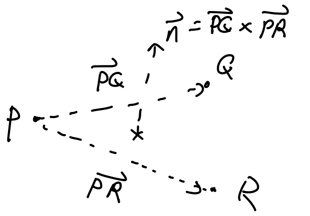
If $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis of $\mathbb{R}^3$:

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$
$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$$
$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

* Cross product is not associative

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta)$$

To get normal vector to a plane:



Hyperplane:

$$\vec{H} = \vec{P} + t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_n\vec{v}_n \quad \text{for } \vec{v}_n \in \mathbb{R}^n \text{ and } \{\vec{v}_1, \dots \vec{v}_n\} \text{ is L.I.}$$

Subspace Test:

Let S be a non-empty subset of $\mathbb{R}^n$. If $\vec{x}+\vec{y} \in S \wedge c\vec{x} \in S$ $\forall \vec{x}, \vec{y} \in S \wedge c \in \mathbb{R}$, then S is a subspace of $\mathbb{R}^n$

Prove:

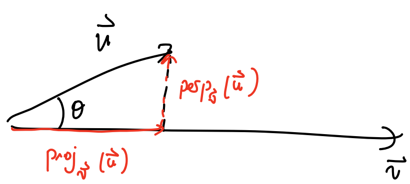
① $\vec{0} \in S$

② $p\vec{x} + q\vec{y} \in S$, $p, q \in \mathbb{R}$

Projection:

$$\text{Proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{Proj}_{\vec{v}}(\vec{u})$$



Systems of Linear Equations

Linear Equation

In the form $a_1 x_1 + \dots + a_n x_n = b$

Coefficient (a_1) variable (x_1) Constant (b)

The linear equation is <u>homogeneous</u> if the constant is $\emptyset$

Systems of linear equations

a set of m equations on n variables

$$\begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n = b_m \end{cases}$$

$\vec{S} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{R}^n$ is a solution if $\vec{S}$ satisfies all equations

$\Rightarrow$ Consistent if there is a solution. Inconsistent otherwise

$\Rightarrow$ Geometrically: m hyperplanes in $\mathbb{R}^n$ $\Rightarrow$ solution is the point/line/plane of intersection.

Solving Systems of Equations

① Gauss - Elimination Method

operations: multiply, interchange, add 2 equations

Note: the first variable w/ non-zero coefficient is leading variable

2 Systems of equations w/ the same solutions are equivalent

② Augmented Matrix

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases} \Rightarrow m \text{ rows} \left\{ \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left| \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix} \right] \right.$$

- Elementary row operations (ERO)
  - Multiply, interchange, add
  - We say M is equivalent to N $(M \sim N)$ if N can be obtained from ERO on M.

Row Echelon Form (REF)

Let A be a $m \times n$ matrix. A is in REF if:

1) Zero rows (if any) is at the bottom
2) The leading coefficient is 1
3) When two non-zero rows are compared, the leading element in the upper row is always to the left of the leading element at lower row.

$\Rightarrow$ The variable corresponding to non-pivot column in REF are free variables

Reduced Row Echelon Form (RREF)

A matrix is in RREF if:

1) A is in REF
2) Every leading element is the only entry in its column

Gauss - Jordan Method

1) put systems in AM
2) Reduce to RREF with ERO
3) Rewrite the equations

Rank

Rank (A) is the # of pivot columns in RREF.

Structure of Solution set

Systems Rank Theorem

- Solution is <u>unique</u> iff $\text{Rank}([A|\vec{b}]) = \text{Rank}([A]) = n$
- The system is <u>consistent</u> iff $\text{Rank}([A|\vec{b}]) = \text{Rank}([A])$
- If the system is consistent:

$$\boxed{\#\text{ of free variables} = n - \text{Rank}([A])}$$

- The system is <u>inconsistent</u> iff $\text{Rank}([A]) < \text{Rank}([A|\vec{b}])$
- $\text{Rank}([A]) = m$ iff the system is consistent for every $\vec{b} \in \mathbb{R}^m$

Span Theorem

- A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ span $\mathbb{R}^n$ iff $\text{Rank}([A]) = n$
  for $t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k = \vec{b}$
- $\{\vec{v}_1, \dots, \vec{v}_k\}$ spans $\mathbb{R}^n$ then $\boxed{k \geq n}$

Linearly Independence Theorem

- A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\} \in \mathbb{R}^n$ is linearly independent iff $\text{Rank}([A])$ for
  $t_1\vec{v}_1 + \dots + t_k\vec{v}_k = \vec{0}$ is n
- If $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent, then $\boxed{k \leq n}$

Basis Theorem

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_{\underline{n}}\}$ is a basis for $\mathbb{R}^n$ iff $\text{Rank}([A]) = n$

Solution Space

If W is a subspace of $\mathbb{R}^n$ with all solutions of the system $[A \mid \vec{0}]$

- **Dimension of Solution Space Theorem**

  If RREF of $[A]$ has "ℓ" free variables, then:

  $$\boxed{\dim(W) = \ell.}$$

- **Solution Theorem**

  For $[A \mid \vec{b}]$ and $[A \mid \vec{0}]$, if W is a solution space for $[A \mid \vec{0}]$ and $\vec{c}$ is any solution to $[A \mid \vec{b}]$,

  then all solutions to $[A \mid \vec{b}]$ is

  $$\boxed{\vec{c} + \vec{w} \quad \text{s.t. } \vec{w} \in W}$$

Matrix

- $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n}$

- $A = B$ iff $(A)_{ij} = (B)_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$

- $M_{m\times n}(\mathbb{R}) \longrightarrow$ collection of all $m\times n$ matrices st. all entries are in $\mathbb{R}$
  $M_{m\times n}(\mathbb{C}) \longrightarrow$ " " " all entries are in $\mathbb{C}$

- Square matrix if $m = n$
- Zero matrix : all entries are zero
- Triangular matrix

  $A \in M_{n\times n}(\mathbb{R}/\mathbb{C})$ st.

  - $a_{ij} = 0$ for $i > j$ $\Rightarrow$ upper triangular
  - $a_{ij} = 0$ for $i < j$ $\Rightarrow$ lower triangular

- Diagonal Matrix

  $D \in M_{n\times n}(\mathbb{R}/\mathbb{C})$ st. $d_{ij} = 0$ $\forall$ $i \neq j$.

- Identity Matrix

  $I \in M_{n\times n}(\mathbb{R})$ $\vee$ $I = \mathrm{diag}(1, 1, \ldots, 1)_n$

Matrices and Vector Space

$$\dim(M_{m\times n}(\mathbb{R})) = m\times n$$

Transpose

If $\vec{x} \in \mathbb{R}^n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$, then $\vec{x}^T = [x_1 \ \dots \ x_n]_{1 \times n} \notin \mathbb{R}^n$

$\Rightarrow$ We can represent a matrix $A \in M_{m \times n}(\mathbb{R})$:

$$A = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix}_{m \times 1}, \quad \vec{a}_i \in \mathbb{R}^n$$

$$= [\vec{a}_1{}' \ \ \vec{a}_2{}' \ \dots \ \vec{a}_n{}'], \quad \vec{a}_i{}' \in \mathbb{R}^m.$$

For a matrix :

$$(A^T)_{ij} = (A)_{ji}$$

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(SA)^T = S A^T$

Matrix Multiplication

- If $A \in M_{m\times n}(\mathbb{R})$ and $\vec{x} \in \mathbb{R}^n$

$$A\vec{x} = \begin{bmatrix} \vec{a}_1^{\,T} \\ \vdots \\ \vec{a}_m^{\,T} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$$

or: $A\vec{x} = \begin{bmatrix} \vec{a}_1{}' & \dots & \vec{a}_n{}' \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1{}' + \dots + x_n \vec{a}_n{}'$

- If $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times c}(\mathbb{R})$

$$(AB)_{ij} = \vec{a}_i \cdot \vec{b}_j$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} B_{11} & \dots & B_{1c} \\ \vdots & \vdots & \vdots \\ B_{n1} & \dots & B_{nc} \end{bmatrix} = \begin{bmatrix} \square_{11} & \dots & \square_{1c} \\ \vdots & \vdots & \vdots \\ \square_{m1} & \dots & \square_{mc} \end{bmatrix}$$

- Properties:

$A\cdot(B+C) = AB + AC$

$(A+B)C = AC + BC$

$s(AB) = (sA)B = A(sB)$

$A(BC) = (AB)C$

Yet:

$AB \neq BA$

$AB = AC$ does not imply $B = C$

$AB = 0$ does not imply $A = [0]$ or $B = [0]$

Block Multiplication

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

Then $AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$

Foudamental Subspaces

Let $A \in M_{m \times n}(\mathbb{R})$

Column Space

$$\begin{aligned} \operatorname{col}(A) &= \text{a subspace of } \mathbb{R}^m \\ &= \operatorname{span}\{\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_n'\}. \\ &= \{A\vec{x} \mid x \in \mathbb{R}^n\}. \end{aligned}$$

$\dim(\operatorname{col}(A)) = \operatorname{Rank}(\operatorname{col}(A)) = \operatorname{rank}(A) = \operatorname{rank}(A^T)$

Null Space

$$\begin{aligned} \operatorname{null}(A) &= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \\ &= \text{a subspace of } \mathbb{R}^n. \end{aligned}$$

- Rank-Nullity Theorem

$$\boxed{\operatorname{rank}(A) + \dim(\operatorname{null}(A)) = n}$$

and $\operatorname{rank}(A) + \dim(\operatorname{null}(A^T)) = m$

Row Space

$$\text{Row}(A) = \text{Col}(A^T) \longrightarrow \text{subspace of } \mathbb{R}^n$$

$$\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = \text{rank}(A) = \text{rank}(A^T)$$

Left-Null Space

$$\text{LNull}(A) = \text{Null}(A^T) \longrightarrow \text{subspace of } \mathbb{R}^m.$$

Inverse of Matrix

Let $A \in \underbrace{M_{n\times n}}_{\text{square}}(\mathbb{R})$. If there exist a $B \in M_{n\times n}(\mathbb{R})$ st.

$$AB = BA = I$$

Then $B = A^{-1}$

↳ If A is invertible, then A^{-1} is unique

To find the inverse

① Consider $[A \mid I]$, reduce to RREF

② If RREF is in the form of $[B \mid I]$, then $B = A^{-1}$

otherwise, A is not invertible.

★ If A is invertible, then:

$$\boxed{\text{rank}(A) = \text{rank}(A^{-1}) = n} \longrightarrow A \text{ is full ranked}$$

and $[A \mid \vec{b}]$ has a unique solution $\vec{x}$ $\Rightarrow$ $\boxed{\vec{x} = A^{-1}\vec{b}}$

Theorems for Invertible Matrices

If $A, B \in M_{n \times n}(\mathbb{R})$ are invertible and $t \in \mathbb{R}$:

- $(tA)^{-1} = \frac{1}{t} A^{-1}$
- $(AB)^{-1} = B^{-1} A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$

- ### Invertible Matrix Theorems

If $A \in M_{n \times n}(\mathbb{R})$ is a invertible matrix, the following conditions are equivalent:

1) A is invertible
2) $\operatorname{rank}(A) = n$
3) $A \overset{RREF}{\sim} I$
4) For $b \in \mathbb{R}^n$, $[A \mid \vec{b}]$ is consistent & has a unique solution
5) $\operatorname{null}(A) = \{\vec{0}\}$
6) $\operatorname{col}(A) = \mathbb{R}^n$
7) $\operatorname{row}(A) = \mathbb{R}^n$
8) $\operatorname{null}(A^T) = \{\vec{0}\}$
9) A^T is invertible
10) Eigenvalue is not 0
11) $\det(A) \neq 0$

Linear Transformation

A function L st. $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

domain: $\mathbb{R}^n$, co-domain: $\mathbb{R}^m$.

1) $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$

2) $L(c\vec{x}) = c\,L(\vec{x})$

3) $L(\vec{0}) = \vec{0}$

* If $L: \mathbb{R}^n \to \mathbb{R}^n$, L is a linear operator

Matrix to L.T

Let $A \in M_{m\times n}(\mathbb{R})$. Then for $f_A: \mathbb{R}^n \to \mathbb{R}^m$

$$\boxed{f_A(\vec{x}) = A_{m\times n}\,\vec{x}_{n\times 1}}$$

L.T. to Marix

If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then

$$\boxed{[L] = [L(\vec{e}_1) \quad L(\vec{e}_2) \quad \dots \quad L(\vec{e}_n)]_{m\times n}}$$

where $\vec{e}_i \in \mathbb{R}^n$ and is the standard basis of $\mathbb{R}^n$

Subspaces

$\mathrm{Null}(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\mathbb{R}^n) = \vec{0}\}$ → Subspace of $\mathbb{R}^n$

$= \text{kernel}$

$\mathrm{Range}(L) = \{L(\vec{x}) \in \mathbb{R}^m \mid x \in \mathbb{R}^n\}$ → subspace of $\mathbb{R}^m$

Determinants

- We can define $\boxed{\det(A) = |A| = ad - bc}$ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

and $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- For any $A \in M_{\underset{\text{square}}{n \times n}}(\mathbb{R})$, let A_{ij} be the $(n-1) \times (n-1)$ sub-matrix of A by deleting the i^{th} row & j^{th} column, and the <u>co-factor</u> is

$$\boxed{C_{ij} = (-1)^{i+j} \det(A_{ij})}$$

Then <u>det (A)</u> is

$$\boxed{\det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}}$$

Can expand on any row/column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & - & - & - & a_{mn} \end{bmatrix}$$

- The matrix is <u>singular</u> if $\det(A) = 0$

- In general,

$$\boxed{A^{-1} = \frac{1}{\det(A)} C^T}$$

where $C = [C_{ij}]$ and $C_{ij} = (-1)^{i+j} |A_{ij}|$

<u>Properties</u>:

1) $\det(AB) = \det(A)\det(B)$

2) $\det(I) = 1$

3) $\det(A^{-1}) = \frac{1}{\det(A)}$

EOR

Let $A \in M_{n \times m}(\mathbb{R})$

i) if 2 rows of A are interchanged, the determinante is negated

ii) if a row or column is scaled by c, then the determinante is also scaled by c.

iii) If a row / column is added to another row / column of A, the determinante does not change

Remarks:

- If A has a row / column of ϕ, then $\det(A) = 0$
- If A is a triangular matrix, then $\det(A)$ is the product of diagonal elements
- If A has two identical rows / columns, then $\det(A) = 0$
  - ↳ If A has a row / colum that is a scalar multiple of another, $\det(A) = 0$
    - ↳ $\boxed{\text{If } A \text{ is not full rank, } \det(A) = 0}$

Eigenvalues and Eigenvectors

Let $A \in M_{n \times n}(\mathbb{R})$. If there exists a non-zero $\vec{v}$ s.t.

$$\boxed{A\vec{v} = \lambda \vec{v}}, \quad \lambda \in \mathbb{R}$$

Then λ is the eigenvalue & $\vec{v}$ is the eigenvector of A.

↳ For a single eigenvalue, there are infinitely many eigenvectors

Finding λ and $\vec{v}$

$\lambda \in \mathbb{R}$ is an eigenvalue of A iff

$$\boxed{\det(A - \lambda I) = 0} \longrightarrow$$

characteristic equation $c(\lambda)$:

$$c(\lambda) = \det(A - \lambda I)$$

$\vec{v}$ is an eigenvector if $\vec{v} \neq \vec{0}$ and

$$\boxed{(A - \lambda I)\vec{v} = \vec{0}}$$

Eigenspace (E_λ)

For a particular λ, the set of all corresponding eigenvectors $+ \vec{0}$ form a subspace of $\mathbb{R}^n$ (for $A \in M_{n \times n}(\mathbb{R})$). This subspace is the eigenspace

Multiplicity:

- Geometric Multiplicity (G.M): $\dim(E_\lambda)$
- Algebraic Multiplicity (A.M): # of times that a λ is repeated

$1 \leq G.M \leq A.M$

(hence if $A.M = 1$, $G.M = 1$)

Diagonalization

$A \in M_{n \times n}(\mathbb{R})$ is diagonalizable if:

1) there is an invertible matrix P
2) there is a diagonal matrix D

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ s.t. $\boxed{P^{-1}AP = D}$

Finding P and D

1) Find $\underline{\lambda}$ of A

2) Find $\underline{\text{basis}}$ for each E_λ

3) If $\sum_{\text{all } E_\lambda}$ basis for each $E_\lambda = n$, A is diagonalizable

   ↳ Basis are column vectors of P

   ↳ λ are diagonal elements of D

Properties

- For $P^{-1}AP = D$, $D^k = P^{-1}A^kP$, $k > 0$

  and $\boxed{A^k = PD^kP^{-1}, \quad k > 0}$

- For $P^{-1}AP = D$ ...

  (1) $\det(A) = \det(D)$

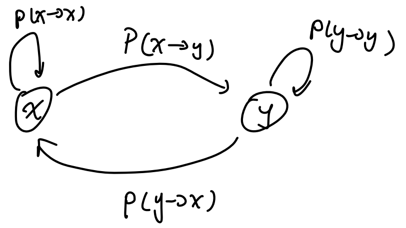
  (2) Same characteristic equations and $\underline{\underline{\text{eigenvalues}}}$

  (3) $\text{rank}(A) = \text{rank}(B)$

  (4) $\text{trace}(A) = \text{trace}(B)$, where:

  $\text{trace}(A) = a_{11} + a_{22} + \dots + a_{nn}$

Markov Process



- **Markov Matrix**

$$\begin{bmatrix} \underbrace{\begin{matrix} p(x\to x) \\ p(x\to y) \end{matrix}}_{\text{Initial state}} & \begin{matrix} p(y\to x) \\ p(y\to y) \end{matrix} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}$$

Require:

- All entries > 0
- Σ each column $= 1$

- For the n^{th} iteration:

$$M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

- M always has an eigenvalue of $\underline{\underline{1}}$

↳ The eigenvector is the <u>steady</u> <u>state</u> of the system

Abstract Vector Space

Basis:

the type of constant for scalar multiplication determines the basis:

ex: For $\mathbb{R}^2$ over $\mathbb{R}$ → basis = $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

For $\mathbb{R}^2$ over $\mathbb{C}$ → basis = $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix} \right\}$

Coordinate for non-standard basis

write the result as a linear combination of all basis

Standard Matrix with non-standard basis

perform linear transformation, then write the result as a linear combination